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MATHEMATICAL MODELING FOR DISSOLVED OXYGEN SAG ANALYSIS IN RIVER

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ABSTRACT

In this paper, we consider the general mass conservation equation, which averaged over the cross section of the stream. We apply this concept for a continuous discharge of wastewater into a river, so which is applicable to the BOD, which also undergoes a first-order decay process. For dissolved oxygen, the one-dimensional mass conservation equation is easily deducted, The time derivative term is removed because steady-state concentrations are sought, and the dispersion term is also removed because of negligible effects for a continuous discharge and appropriate source and sink terms are added and the maximum dissolved-oxygen deficit is obtained. We also discussed zero order kinetics model, derived on the basis of assumptions that the concentration of bacteria is relatively constant and that the rate of substrate utilization is zero order. We find D_t for the dissolved oxygen deficit at the transition when the substrate is all utilized. For times greater than $t = \frac{L_0}{k}$ there is no longer any substrate and thus $k_0 = 0$.

KEYWORDS: *Mass Conservation Equation, Dissolved Oxygen, Steady-State, BOD, Continuous Discharge.*

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1. INTRODUCTION

The discharge of a sewage effluent in a stream produces a biochemical oxygen demand (BOD) which decays exponentially in time and space. This oxygen demand causes an oxygen deficit, or oxygen shortage. The greater oxygen deficit, greater the rate of natural oxygen replenishment from the atmosphere into the stream. These two concurrent processes of oxygen consumption and oxygen replenishment produce an oxygen sag curve, i.e., a curve which sags initially due to the increased oxygen demand and recovers asymptotically downstream due to the increased rate of oxygen replenishment.

Dissolved oxygen (DO), the most significant health index of a river necessitates an accurate prediction. The traditionally used Streeter–Phelps (1925) models for DO sag not accounting for the settle able biochemical oxygen demand (BOD) (constituting a very significant portion of the total BOD inputs through the wastewater outfalls into a river), can therefore, no longer be used for an accurate and rational prediction of a stream's DO. To account for the stated removal through sedimentation of the settle able BOD, Streeter (1935) realized its consideration; Thomas (19401948) considered (though less elegantly) this effect through a 'lag-time'; Orford and Ingram (1953) introduced
a coefficient of retardation in their models; and Velz and Gannon (1962) took account of the BOD removals through formation and settling of biological flocs by adding a factor a to the BOD rate constant in the exponential form of a first-order kinetics which was not rational or justified at all because the flocs removal through sedimentation can continue only for a short limited striking distance (or time) in a river after which distance, the effect of their added a factor will not be valid due to its incorporation (exponentially) in their first-order rate constant. Bhargava (1983, 1986a) for the first time, presented a rational composited model for stream BOD prediction accurately, and this model takes account of the near linear removal of the settle able part of the BOD through its bioflocculation and sedimentation as well as a D. S. Bhargava Roorkee University (now IIT), Roorkee, UA, India D. S. Bhargava AIT Bangkok, P.O. Box 4, Klong Luang, Pathum Thani, Thailand D. S. Bhargava (&) Bhargave Lane, Devpura, Haridwar, simultaneous first-order exponential decay of the non- settle able and/or dissolved portion of the BOD. Based this compo sited BOD predictive model, the stream DO sag parameters were evolved for a rational and accurate pre-diction of the stream's DO. A detailed account of such models for DO sag related parameters has been presented and tested with observed data to justify the robustness of the evolved models, elsewhere, for a simple case of only one wastewater outfall (Bhargava 1986a) and also for multi-wastewater outfalls spaced at intervals >T (the time during which the near linear removal of the settleable BOD is almost completed) such

that the settling fields of one outfall does not interfere (or overlap) with the settling field of the next downstream outfall (Bhargava1986b).

However, in many situations, the outfalls may very closely be spaced at intervals <T, wherein the settling field distance of one outfall would interfere (or overlap) with the settling field distance of the next downstream outfall(s). The multi-outfalls system, wherein the outfalls are so spaced that some of them interfere their settling fields and others do not interfere, is chosen to analyse a stream's DO sag. The settle able portion of the BOD accumulates in the stream bottom to form benthel sludge. There may be some relatively insignificant contribution of non-settle able BOD by the benthal sludge through leaching into the continuous flowing river water. The observed BODs which were almost equal to the evaluated BODs (through mass balancing) indicate that the benthic loading was insufficient. It may however, be of some significance in very shallow rivers and open drains [4].

2. ONE-DIMENSIONAL MODELING

Rivers and estuaries are generally many times longer than they are wide or deep. As a result, inputs from wastewater treatment plants or other sources rapidly mix over the cross section, and a one – dimensional approach is often justified. In the one-dimensional approach, only longitudinal variations of constituent concentrations are resolved in the form of cross –section- averaged values. The general mass conservation equation is averaged over the cross section of the stream giving, for constituents subjects to a single first-order decay-process [6],

$$\frac{\partial C}{\partial t} = -U\frac{\partial C}{\partial x} + \frac{\partial}{\partial x}\left[(D_x + D_L)\frac{\partial C}{\partial x}\right] - KC + \Sigma I \qquad (2.1)$$

Where, x=longitudinal distance along river or estuary, L

D_L=longitudinal dispersion coefficient, L^2/T

Equation 2.1 is almost identical to conservation of mass equation without terms containing *y and z* derivatives, except for the appearance of dispersion, which is distinct and separate from turbulent diffusion. The dispersion term arises during the averaging process (which is somewhat involved mathematically) due to the correlation of cross-sectional velocity and concentration variations [5, 15]. Dispersion in natural streams is predominantly due to lateral velocity variations, and the following formula can be used to estimate coefficient

[5]:

$$D_L = 0.011 \frac{U^2 B^2}{Hu^*}$$

(2.2)

Where D_L=longitudinal dispersion coefficient, L^2/T

U=Cross- section-averaged velocity, L/T

B= Stream width, L

H= stream depth, L

u^* =shear velocity, L/T =$\sqrt{\text{gHs}}$

g =acceleration due to gravity, L^2/T

s=stream slope, L/L

Equation 2.2 remains approximate because it does not account for dead zones in which matter can get trapped, thereby increasing the effective dispersion coefficient. Bends can increase or decrease dispersion depending on their configuration; in particular, successive bends can increase dispersion if their separation is small. In estuaries, tidal flow reversals as well as secondary currents driven by salinity gradients tend to increase dispersion [8]. Dispersion is typically much larger than turbulent diffusion so that D_x can be neglected, compared to D_L in Eq. 2.1.

3. CONTINUOUS DISCHARGE.

The solution of Eq. 3.1 for continuous discharge at $x = 0$ is [9].

$$C = \frac{M'}{A\sqrt{U^2 + 4KE_L}} e^{(xU/2D_L)\,(1 \pm \sqrt{1+4\,D_L/U^2}}$$

(3.1)

Where M'=Discharge rate, M/T =$Q_D C_D$

Q_D=Discharge flowrate, L^3/T

C_D=discharge concentration, M/L^3

$\pm$= + for $x < 0$ and $-$ for $x > 0$

Note that for a conservative substance (K=0) the concentration is uniform and equal to M'/AUdownstream of the discharge point. The upstream intrusion is not greatly affected by the decay coefficient. In many cases, the value of the term $4KD/U^2$is small compared to others. For example, the term $4KD_L/u$ equals 0.0028 for the following typical values: U=1 ft/s, K=0.30 d^{-1} =3.5 X $10^{-6}s^{-1}$, and D_L =200 ft^2/s. In this case, concentrations downstream of the source are very closely given by [10, 2]

$$C = \frac{M'}{AU} e^{-Kx/U}; (x > 0)$$

(3.2)

This is independent of the dispersion coefficient. Thus, it is generally true that dispersion can be neglected for continuous discharges in rivers [7].

4. SAG ANALYSIS

First, we develop a relationship for the change in oxygen concentration due to oxidation of organics. The rate that oxygen is used will be proportional to the rate that substrate (or biochemical oxygen demand) is oxidized. The rate of substrate utilization by bacteria is given by the Monod relationship [4]

$$\frac{dL}{dt} = \frac{-k}{K_s}$$

(4.1)

Where L substrate concentration is expressed as oxygen demand or BOD [mg/L], k is the maximum specific substrate utilization rate, K_s is the half velocity constant, and X is the concentration of bacteria. However, the concentration of bacteria is a function of the substrate concentration and thus application of the Monod equation to a polluted river is not trivial. Often the bacterial concentration remains relatively constant. If the half velocity concentration is large relative to the concentration of substrate we obtain [3]

$$L = L_o e^{-Kt}$$

(4.2)

We are interested in the oxygen deficit as a function of distance down a stream. As an approximation we can think of a cross section of a river as a completely stirred reactor that is slowly moving downstream. The relation between time in a batch reactor and distance down the river is simply $t =$

The above concept we can apply for a continuous discharge of wastewater into a river, so Eq. 3.2 is applicable to the BOD, which also undergoes a first-order decay process. So equation 3.2 reduces to

$$L = L_o e^{-Kx/U}$$

(4.3)

Where $L_o = M'/AU$. This approach is valid for both carbonaceous BOD and for nitrogenous BOD, provided that x is referenced to the point where nitrogenous BOD begins to be exerted. Thus, nitrogenous BOD can be considered as a fictitious source of BOD at a point downstream of the real source, separated by the travel time, t= x/U, Equal to the nitrogenous delay time [6].

Where$U$ is the stream velocity and x is distance. The Streeter-Phelps model assumes a constant input of biodegradable substrate, L_o, at $x = 0$ and the model is valid under steady-state conditions. The maximum oxygen deficit occurs when [4]

$$\frac{\partial D}{dt} = 0$$

(4.4)

For dissolved oxygen, the one-dimensional mass conservation equation is easily deducted from Eq. 2.1.The time derivative term is removed because steady-state concentrations are sought, and the dispersion term is also removed, as it was shown above to have negligible effects for a continuous discharge. The appropriate source and sink terms are added from $\sum r_i = -r_0 + r_R - r_s + r_p - r_{Rp}$ [11, 12]

where

r_p = rate of oxygen production by photosynthesis per unit time of water per unit volume of water, M/TL^3

r_{Rp} = rate of oxygen consumption due to respiration per unit time of water per unit volume of water, M/TL^3, gives

$$O = -U\frac{dC}{dx} - KL + K_2(C_s - C) + \left(r_p - r_{R_p} - \frac{k_s}{H} + \Sigma I\right)$$

(4.5)

The solution of this equation for a continuous discharge at $x = 0$ is [17]

$$D = \frac{KL_0}{K_2 - K}\left(e^{-Kt} - e^{-K_2 t}\right) + D_0 e^{-K_2 t} - \frac{r_p - r_{Rp} - k_s/H}{K^2}\left(1 - e^{-K_2 t}\right)$$

(4.6)

Where

D=dissolved-oxygen deficit =$C_S - C$

r_R = rate of oxygen gain due to reaeration per unit time per unit volume of water , M/TL^3

C_S= Saturation dissolved-oxygen concentration, M/L^3

C= Dissolved-oxygen concentration, M/L^3

K_2 = Surface reaeration rate, 1/T

D_0=Dissolved-oxygen deficit at x=0

t= travel time = x/U

Because Eq. 4.6 is general, it can be applied to different situations. For example, if photosynthesis, respiration, and sediment oxygen demand are not significant, the last term in the equation is zero and can be removed, we get [19]

$$D = \frac{K L_0}{K_2 - K}\left(e^{-Kt} - e^{-K_2 t}\right) + D_0 e^{-K_2 t}$$

(4.7)

In this case, eq.4.6 gives the classic sag curve [14]. Downstream of the discharge point, BOD exertion results in a decrease of dissolved oxygen. Concurrently, dissolved oxygen is replenished through surface reaeration at a rate proportional to the DO deficit. At a certain distance from the discharge point, the input from reaeration equals the BOD consumption and the DO deficit reaches a maximum. Downstream of this point, input exceeds consumption and the deficit decreases. The point of maximum dissolved-oxygen deficit is obtained by differentiating eq. 4.6 with the respect to travel time and setting the derivative to zero which yields [4]

$$t_{max} = \frac{1}{K_2 - K} ln \left[\frac{K_2}{K}\left(1 - \frac{-D_0(K_2 - K)}{K L_0}\right)\right]$$

(4.8)

and

$$D_{max} = \frac{K}{K_2} L_0 e^{-K t_{max}}$$

(4.9)

Equation 4.6 is valid for a stretch of river without sources or tributaries. It can however, be applied sequentially for streatches between sources and tributaries by adjusting L_0 and D_0 for each stretch

5. ZERO ORDER KINETICS MODEL

An alternate model can be derived based on the assumptions that the concentration of bacteria is relatively constant and that the rate of substrate utilization is zero order, i.e., the concentration of the substrate is greater than the half velocity constant K_S.

$$\frac{dL}{dt} = \frac{-kLX}{K_s + L} \cong -kX = -$$

(5.1)

The change of the deficit of dissolved oxygen is equal to the change caused by microbial degradation ( ) plus change due to endogenous respiration minus the reaeration (k_2).

$$\frac{dD}{dt} = k_0 + k_s - k_2 D$$

(5.2)

This equation is only valid when the substrate concentration is greater than K_s. To simplify derivation, assume that K_s is very small relative to the initial BOD added to the system and

apply the zero-order model until the substrate is completely oxidized. When the substrate concentration reaches zero a discontinuity will occur as substrate oxidation stops. Separating variables and integrating

$$\int_{D_0}^{D} \frac{dD}{k_0 + k_e - k_2 D} = \int_0^t dt$$

implies

and solving for the dissolved oxygen deficit

(5.3)

The substrate concentration is depleted when $t = \frac{L_0}{k_0}$. Substituting into equation 5.3 to get the maximum dissolved oxygen deficit yields [4]

$$D_t = \frac{k_0 + k_e(k_0 + k_e - k_2 D_0)e^{\frac{L_0}{k_0}}}{k_2}$$

(5.4)

where D_t is the dissolved oxygen deficit at the transition when the substrate is all utilized. For times greater than $t = \frac{L_0}{k}$ there is no longer any substrate and thus $k_0 = 0$ and equation 5.4 becomes

$$\frac{dD}{dt} = k_e - k_2 D$$

[5.5]

Separating variables and integrating equation 5.5, we get

$$\int_{D_t}^{D} \frac{dD}{k_e - k_2 D} = \int_{t_t}^{t} dt$$

$$\frac{-1}{k_2} ln \frac{k_e - k_2 D}{k_e - k_2 D_t} = t - t_t$$

$$D = \frac{k_g(k_g - k_2 D_t)e^{k_2(t_t}}{k_2}$$

[5.6]

Equation 5.6 is valid for all times greater than [1].

CONCLUSION

The downstream of the discharge point, BOD exertion results in a decrease of dissolved oxygen. Concurrently, dissolved oxygen is replenished through surface reaeration at a rate proportional to the DO deficit. At a certain distance from the discharge point, the input from reaeration equals the BOD consumption and the DO deficit reaches a maximum. Downstream of this point, input exceeds consumption and the deficit decreases. The point of maximum dissolved-oxygen deficit is obtained by differentiating eq. 4.6 with the respect to travel time and setting the derivative to zero and in zero order kinetics model

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